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Fundamental Systems of Units in Biquadratic Parametric Number Fields*

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Let D, d be integers with $D > 0$, d dividing D and d square free and let α and β be the two roots of the quadratic equation $x^2 - Dx - d = 0$. Suppose $(\alpha - \beta)^2 = D^2 + 4d > 1$ and $\alpha^2 + \beta^2 = D^2 + 2d$ be square free. Introduce $e = (\omega + \alpha)/\beta$ and $u = (\omega + \beta)/\alpha$, where $\omega = \sqrt{\alpha^2 + \beta^2}$. Then it is proved that e, u and θ form a fundamental system of units in the field $\mathbb{Q}(\sqrt{D^2 + 4d}, \sqrt{D^2 + 2d})$, where

$$\theta = \frac{\alpha}{\beta} \qquad \text{if } d \neq \pm 1 \text{ and } (D, d) \neq (5, -5)$$

$$\theta = \sqrt{\frac{\alpha}{\beta}} = \frac{1 + \sqrt{5}}{2} \qquad \text{if } (D, d) = (5, -5)$$

$$\theta = \alpha \qquad \text{if } d = \pm 1 \text{ and } (D, d) \neq (3, -1)$$

$$\theta = \sqrt{\alpha} = \frac{1 + \sqrt{5}}{2} \qquad \text{if } (D, d) = (3, -1).$$

0. Introduction

Let D and d be integers with $D > 0$, d dividing D and d square free and suppose that $M = D^2 + 4d > 1$. Let us introduce α and β as the two distinct roots of the quadratic equation $X^2 - DX - d = (X - \alpha)(X - \beta)$; that is, $\alpha + \beta = D$, $\alpha \cdot \beta = -d$ and $(\alpha - \beta)^2 = D^2 + 4d = M$.

Furthermore we put $\delta = \alpha - \beta = \sqrt{D^2 + 4D} = \sqrt{M}$, $\omega = (\alpha^n + \beta^n)^{1/n} = M_n^{1/n}$ where $M_n = \alpha^n + \beta^n = \omega^n > 1$, and

$$e_{n,k} = \frac{\omega^k - \alpha^k}{\beta^k}, \qquad u_{n,k} = \frac{\omega^k - \beta^k}{\alpha^k}, \qquad \eta = \frac{\alpha}{\beta}.$$

Then we proved in [1] that the set

$$S = S(L_{2n}) = \{e_{n,k}, u_{n,k}, \eta \mid k \in \mathbb{N}, k \mid n, k \neq n\}$$

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is an independent system of units in the real algebraic number field $L_{2n} = K_n(\delta) = Q(\omega, \delta)$ of degree $2n$ over Q. Note that our η is the reciprocal of the η in [1] and that $K_n = Q(\omega)$.

$S(L_{2n})$ is a full independent system if $n = 1, 2, 3, 4, 6$. In this paper we shall investigate when the system S is a fundamental system of units for the cases $n = 1, 2$.

1. The Case $n = 1$: $L_2 = Q(\sqrt{D^2 + 4d})$

For the field $L_2 = Q(\sqrt{D^2 + 4d}) = Q(\alpha) = Q(\beta) = Q(\delta)$ we have $S(L_2) = \{\eta\}$, where

$$\eta = \frac{\alpha}{\beta} = \frac{(\alpha + \beta) + (\alpha - \beta)}{(\alpha + \beta) - (\alpha - \beta)} = \frac{D + \delta}{D - \delta} = \frac{(D + \delta)^2}{D^2 - \delta^2} = \frac{(D + \delta)^2}{-4d}.$$

We shall apply the following theorem due to H.-J. Stender (see Satz 11 in [3]), which generalizes a result of Degert.

THEOREM 1. *Let $c = 1, 2$ or 4 and let r, R be natural numbers with*

$$(c, R) = 1, \qquad r \textit{ square free and } r \mid R;$$

furthermore let $T = R^2 \pm cr > 1$ and $\tau = \sqrt{T}$.

If T or R^2/r is square free, then

$$\zeta = \frac{R + \tau}{\sqrt{cr}} \qquad \textit{in the case } c = 1, 4 \textit{ and if in addition}$$

$$r = 1 \textit{ or if } \frac{T}{r} \textit{ is a square}$$

$$\zeta = \frac{(R + \tau)^2}{cr} \qquad \textit{otherwise}$$

is the fundamental unit of the real quadratic field $K = Q(\tau)$, except for $(R, cr) = (2, 1)$, resp. $(5, 20)$, in the case of the positive sign (then $T = 5$, resp. 45) and for $(R, cr) = (5, 5)$, resp. $(3, 4)$, in the case of the negative sign (then $T = 20$, resp. 5).

In the case $(R, cr) = (2, 1)$ and positive sign and in the case $(R, cr) = (5, 5)$ and negative sign $\zeta^{1/3}$ is the fundamental unit of K; in all the other exceptional cases $\sqrt{\zeta}$ is the fundamental unit in K.

From this theorem we get

THEOREM 2. *Let $M = D^2 + 4d > 1$ be square free, $d \mid D$ and $D > 0$.*

(a) *Suppose* $d \neq \pm 1$, *then* $\eta = \alpha/\beta$ *is the fundamental unit in* L_2 *if* $d < 0$, *and* $\eta = \alpha/-\beta$ *is the fundamental unit in* L_2 *if* $d > 0$, *except when* $(D, d) = (5, -5)$, *in which case* $\sqrt{\alpha/-\beta} = (1 + \sqrt{5})/2$ *is the fundamental unit in* $L_2 = Q(\sqrt{5})$.

(b) *If* $d = -1$ *then* $\sqrt{\eta} = \alpha$ *is the fundamental unit in* L_2, *and if* $d = +1$ *then* $\sqrt{-\eta} = \alpha$ *is the fundamental unit in* L_2, *except when* $(D, d) = (3, -1)$, *in which case* $\eta^{1/4} = \sqrt{\alpha} = (1 + \sqrt{5})/2$ *is the fundamental unit in* $L_2 = Q(\sqrt{5})$.

Proof. We only indicate that $c \neq 1, 2$, as $M = T$ has to be square free. Hence $c = 4$, $R = D$, $r = |d|$. Furthermore if $M/|d|$ is a square, then $M = D^2 + 4d = -d$ and hence $(D, d) = (5, -5)$.

Remarks: (1) The fundamental unit η_0 in L_2 is by definition so determined that $\eta_0 > 1$. Note that $\eta > 0$ if $d < 0$ and $\eta < 0$ if $d > 0$, and that $|\alpha| > |\beta|$ (see [1]). If η_0 is *the* fundamental unit in a field F we shall call each of the units η_0, $-\eta_0$, η_0^{-1} and $-\eta_0^{-1}$ *a* fundamental unit of F.

(2) Theorem 2 could be slightly extended with the use of the full strength of Theorem 1.

2. The Case $n = 2$: $L_4 = Q(\sqrt{D^2 + 2d}, \sqrt{D^2 + 4d})$

2.1. The real field $L_4 = Q(\omega, \delta)$ contains the three real quadratic subfields $L_2 = Q(\delta)$, $K_2 = Q(\omega)$ and $N_2 = Q(\delta\omega)$, where $\omega^2 = \alpha^2 + \beta^2 = D^2 + 2d = M_2$ and $\delta^2 = (\alpha - \beta)^2 = D^2 + 4d = M$. We shall first determine the fundamental units in L_2, K_2 and N_2, respectively, and then with their help a fundamental system of units in L_4. In view of Theorem 2 we always require $M = D^2 + 4d > 1$ to be square free.

The fundamental unit in L_2 is already determined by Theorem 2. For the field K_2 we set

$$\begin{aligned}\varepsilon = \varepsilon_{2,1} &= \frac{(\omega - \alpha)(\omega - \beta)}{\alpha\beta} = \frac{\omega^2 - \omega(\alpha + \beta) + \alpha\beta}{\alpha\beta} \\ &= \frac{2(\alpha^2 + \beta^2) - 2\omega(\alpha + \beta) + 2\alpha\beta}{2\alpha\beta} \\ &= \frac{\alpha^2 + \beta^2 - 2\omega(\alpha + \beta) + (\alpha + \beta)^2}{2\alpha\beta} \\ &= \frac{\omega^2 - 2\omega D + D^2}{-2d} = \frac{(\omega - D)^2}{-2d},\end{aligned}$$

which is a unit in K_2 (see [1]), and

$$\varepsilon' = \varepsilon^{-1} = \frac{(\omega+\alpha)(\omega+\beta)}{\alpha\beta} = \frac{(\omega+D)^2}{-2d},$$

where ε' is the conjugate of ε.

From Theorem 1 we extract

PROPOSITION 3. *Let $M_2 = D^2 + 2d > 1$ and $M = D^2 + 4d > 1$ be square free, $d \mid D$, and $D > 0$. Then*

ε' is the fundamental unit in K_2 if $d < 0$, and

$-\varepsilon'$ is the fundamental unit in K_2 if $d > 0$.

Again, Proposition 3 could be slightly improved by use of the full strength of Theorem 1.

For the field $N_2 = Q(\nu)$, where

$$\begin{aligned}\nu = \delta\omega &= \sqrt{MM_2} = \sqrt{(D^2+4d)(D^2+2d)}\\ &= \sqrt{D^4 + 6dD^2 + 8d^2} = \sqrt{(D^2+3d)^2 - d^2}\end{aligned}$$

we first notice that

$$\begin{aligned}(M, M_2) &= (D^2+4d, D^2+2d) = (2d, D^2+2d)\\ &= \begin{cases} d & \text{if } 2 \nmid D\\ 2d & \text{if } 2 \mid D.\end{cases}\end{aligned}$$

If M is square free, then $2 \nmid D$ and hence $(M/d, M_2/d) = 1$. So $MM_2/d^2 = ((D^2+3d)/d)^2 - 1$ is square free if and only if M and M_2 are square free. We now set

$$\begin{aligned}\gamma &= \frac{(\omega+\alpha)(\omega-\beta)}{\alpha\beta} = \frac{1}{\alpha\beta}(\alpha^2+\beta^2-\alpha\beta+(\alpha-\beta)\omega)\\ &= \frac{1}{\alpha\beta}((\alpha+\beta)^2 - 3\alpha\beta + \delta\omega) = \frac{1}{-d}(D^2+3d+\nu)\end{aligned}$$

which is a unit in N_2, as $(\omega+\alpha)/\beta$ and $(\omega-\beta)/\alpha$ are units in L_4 (see [1]).

We apply again Theorem 1 and obtain

PROPOSITION 4. *Let $M = D^2 + 4d > 1$ and $M_2 = D^2 + 2d > 1$ be square free, $d \mid D$ and $D > 0$. Then*

γ is the fundamental unit in N_2 if $d < 0$, and

$-\gamma$ is the fundamental unit in N_2 if $d > 0$.

2.2. Henceforth we shall always suppose that $M = D^2 + 4d > 1$ and $M_2 = D^2 + 2d > 1$ are square free. Let θ stand for a fundamental unit in L_2; more precisely

$$
\begin{array}{ll}
\theta = \eta = \dfrac{\alpha}{\beta} & \text{if } d \neq \pm 1 \text{ and } (D, d) \neq (5, -5) \\[2ex]
\theta = \sqrt{-\eta} = \sqrt{\dfrac{\alpha}{-\beta}} = \dfrac{1+\sqrt{5}}{2} & \text{if } (D, d) = (5, -5) \\[2ex]
\theta = \alpha & \text{if } d = \pm 1 \text{ and } (D, d) \neq (3, -1) \\[2ex]
\theta = \sqrt{\alpha} = \dfrac{1+\sqrt{5}}{2} & \text{if } (D, d) = (3, -1).
\end{array}
$$

Let $\langle\sigma\rangle = \mathrm{Gal}(L_4, L_2)$, $\langle\tau\rangle = \mathrm{Gal}(L_4, K_2)$ and $\langle\rho\rangle = \mathrm{Gal}(L_4, N_2)$; that is, σ, τ and ρ are the generators of the Galois groups of L_4 over F for $F = L_2$, K_2 and N_2, respectively. The automorphisms σ, τ and ρ are determined by

$$
\begin{aligned}
\sigma &: (\omega, \delta) \longmapsto (-\omega, \delta), \\
\tau &: (\omega, \delta) \longmapsto (\omega, -\delta), \\
\rho &: (\omega, \delta) \longmapsto (-\omega, -\delta).
\end{aligned}
$$

Hence we have $\sigma(\alpha, \beta) = (\alpha, \beta)$, $\tau(\alpha, \beta) = \rho(\alpha, \beta) = (\beta, \alpha)$ and $\rho = \sigma\tau = \tau\sigma$; furthermore, $\mathrm{Gal}(L_4, Q) = \{\mathrm{id}, \sigma, \tau, \rho\}$.

We now introduce

$$
e = \frac{\omega + \alpha}{\beta} \qquad \text{and} \qquad u = \frac{\omega + \beta}{\alpha}
$$

and we note that

$$
\varepsilon' = e \cdot u \qquad \text{and} \qquad \gamma = -e \cdot u^{\sigma} = e \cdot u^{-1}.
$$

For the convenience of the reader we display the norm relations:

$$
\begin{aligned}
N_{L_4/L_2}(e) &= N_{L_4/L_2}(u) = -1, \\
N_{L_4/K_2}(e) &= N_{L_4/K_2}(u) = \varepsilon', \\
N_{L_4/N_2}(e) &= (N_{L_4/N_2}(u))^{-1} = -\gamma, \\
N_{L_4/L_2}(\theta) &= \theta^2,
\end{aligned}
$$

$$
\begin{aligned}
L_{L_4/K_2}(\theta) = N_{L_4/N_2}(\theta) &= -1 \qquad \text{if } d = +1 \text{ or } (D, d) = (3, -1), (5, -5) \\
&= +1 \qquad \text{otherwise}
\end{aligned}
$$

The index i of the unit group $E' = \langle -1, \varepsilon', \gamma, \theta \rangle$ generated by -1, ε', γ and θ within the total unit group $E(L_4)$ of L_4 is a divisor of 8 (see [2, §7]). Since $E'' = \langle -1, e, u, \theta \rangle$ contains E' but not vice versa, the index of E'' in $E(L_4)$ can only be a divisor of 4. So $E(L_4)$ is generated by -1, e, u, θ and possibly square roots of

$$\pm e, \qquad \pm u, \qquad \pm\theta, \qquad \pm eu, \qquad \pm e\theta, \qquad \pm u\theta, \qquad \pm eu\theta$$

(see [1, Satz 11]).

We are now going to show that these 14 units are all non-squares in L_4 and that therefore

$$E(L_4) = E'' = \langle -1, e, u, \theta \rangle = \langle -1, e^\sigma, u^\sigma, \theta^\sigma \rangle$$
$$= \left\langle -1, \frac{\omega - \alpha}{\beta}, \frac{\omega - \beta}{\alpha}, \theta^\sigma \right\rangle.$$

For that purpose we shall make use of the following lemma of Wada ([4, §1]).

LEMMA 5. (i) *If $A \in L_4$ is a square in L_4, then $N_{L_4/F}(A)$ are squares in F for $F = L_2, K_2$ and N_2.*

(ii) *For $A \in L_4$ let $N_{L_4/F}(A)$ be squares in F for $F = L_2$, K_2 and N_2. Then A is a square in L_4 if and only if one of the rational numbers c, $c \cdot d(L_2 \mid Q)$, $c \cdot d(K_2 \mid Q)$ or $c \cdot d(N_2 \mid Q)$ is a square in Q, where $d(F, Q)$ denotes the discriminant of F over Q,*

$$c = T_{L_4/Q}(\xi) = \xi + \xi^\sigma + \xi^\tau + \xi^\rho, \qquad \xi = B_1 B_2 B_3 + b(B_1 + B_2 + B_3),$$
$$b = \sqrt{N_{L_4/Q}(A)}, \qquad B_1 = \sqrt{N_{L_4/L_2}(A)},$$
$$B_2 = \sqrt{N_{L_4/K_2}(A)}, \qquad B_3 = \sqrt{(N_{L_4/N_2}(A))^\sigma}.$$

PROPOSITION 6. *Let $M = D^2 + 4d > 1$ and $M_2 = D^2 + 2d > 1$ be square free, $d \mid D$ and $D > 0$. Then*

$$\pm e, \qquad \pm u, \qquad \pm\theta, \qquad +eu, \qquad \pm e\theta, \qquad \pm u\theta, \qquad \pm eu\theta$$

are all non-square units in L_4.

Proof. (a) $\pm e$ and $\pm u$ cannot be squares in L_4, for if they were, then also $\varepsilon' = N_{L_4/K_2}(\pm e) = N_{L_4/K_2}(\pm u)$ would have to be a square in K_2. Similarly, $\pm e\theta$ and $\pm u\theta$ cannot be squares in L_4, as otherwise $N_{L_4/K_2}(\pm e\theta) = N_{L_4/K_2}(\pm u\theta) = -\varepsilon'$ in the case $d = +1$ or $(D, d) = (3, -1)$, $(5, -5)$, and $N_{L_4/K_2}(\pm e\theta) = N_{L_4/K_2}(\pm u\theta) = +\varepsilon'$ in the other cases would have to be a square in K_2.

(b) For the remaining six units we infer Lemma 5. For $A = \pm eu = \pm\varepsilon'$ we compute $B_1 = B_3 = b = 1$, $B_2 = \varepsilon'$, $\xi = 2(\varepsilon' + 1)$,

$$c = 4\left(\frac{(\omega+\alpha)(\omega+\beta)}{\alpha\beta} + 1\right) + 4\left(\frac{(\omega-\alpha)(\omega-\beta)}{\alpha\beta} + 1\right)$$

$$= \frac{4}{\alpha\beta}\left[\omega^2 + \omega(\alpha+\beta) + \alpha\beta + \omega^2 - \omega(\alpha+\beta) + \alpha\beta + 2\alpha\beta\right]$$

$$= \frac{4\cdot 2}{\alpha\beta}(\alpha+\beta)^2 = 4D^2 \cdot \frac{2}{-d}.$$

c is not a square, since d is square free and $d \neq -2$ ($d = -2$ would imply $2 \mid D$ and hence M_2 and M not square free). Since

$$d(L_2 \mid Q) = M = D^2 + 4d$$

is square free, $d(K_2 \mid Q) = 4M_2 = 4(D^2 + 2d)$ with $D^2 + 2d$ square free and $d(N_2 \mid Q) = 4MM_2/d^2$ with MM_2/d^2 square free, we get that none of the numbers $c \cdot d(F \mid Q)$ with $F = L_2, K_2, N_2$ can be a square in Q. Just notice that $D^2 + 4d = -2d$, $D^2 + 2d = -2d$ and $(D^2 + 4d)(D^2 + 2d)/d^2 = -2d$ would always imply $2 \mid D$ and hence M not square free.

(c) For $A = \pm\theta$ we have $N_{L_4|K_2}(\pm\theta) = N_{L_4|N_2}(\pm\theta) = -1$ in the case $d = +1$ and in the cases $(D, d) = (3, -1)$, $(5, -5)$, so that $\pm\theta$ cannot be a square in these cases. In the other two cases $d \neq \pm 1$ and $d = -1$ with $D \neq 3$ we compute $B_1 = \theta$, $B_2 = B_3 = b = 1$, $\xi = 2(\theta + 1)$.

(c1) If $d \neq \pm 1$ then $\theta = \alpha/\beta$ and so

$$c = 4\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 2\right) = \frac{4}{\alpha\beta}(\alpha+\beta)^2 = \frac{4D^2}{-d},$$

which cannot be a square; nor are $c \cdot d(F \mid Q)$ squares for $F = L_2$, K_2 and N_2. Just notice that $D^2 + 2d = -d$ would yield $(D, d) = (3, -3)$ and hence $M = D^2 + 4d < 0$; that $D^2 + 4d = -d$ would give $(D, d) = (5, -5)$ and that $(D^2 + 4d)(D^2 + 2d)/d^2 = -d$ would imply $(D^2 + 4d)/d = d_1$ and $(D^2 + 2d)/d = d_2$ with $d_1 d_2 = -d > 0$ and hence $d_1 = d_2 = -1$ and so $D^2 + 4d = -d$ and $D^2 + 2d = -d$ as before.

(c2) If $d = -1$ and $D \neq 3$, then $\theta = \alpha$ and hence

$$c = 4(\alpha + \beta + 2) = 4(D + 2),$$

which is not a square since $M = D^2 - 4 = (D + 2)(D - 2)$ is square free. $c \cdot d(F \mid Q)$ are non-squares for $F = L_2, K_2, N_2$ because $D^2 - 2 = D + 2$ and $(D^2 - 2)(D^2 - 4) = D + 2$ have no integral solutions and $D^2 - 4 = D + 2$ leads to $D = 3$.

(d) For $A = \pm eu\theta$ we have $N_{L_4/N_2}(A) = -1$ if $d = +1$ or if $(D, d) = (3, -1)$, $(5, -5)$, so that A cannot be a square in these cases. For the remaining cases we compute $B_1 = \theta$, $B_2 = \varepsilon'$, $B_3 = b = 1$, $\xi = \theta\varepsilon' + (\theta + \varepsilon' + 1) = (\theta + 1)(\varepsilon' + 1)$, where

$$\varepsilon' + 1 = \frac{(\omega + \alpha)(\omega + \beta) + \alpha\beta}{\alpha\beta} = \frac{\alpha^2 + \beta^2 + \omega(\alpha + \beta) + 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)(\alpha + \beta + \omega)}{\alpha\beta} = \frac{D(D + \omega)}{-d}.$$

(d1) If $d \neq \pm 1$, then $\theta = \alpha/\beta$, hence $\theta + 1 = D/\beta$ and therefore

$$c = \frac{D^2}{-d} T_{L_4/Q}\left(\frac{D + \omega}{\beta}\right) = \frac{D^2}{-d} \cdot 2D\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{2D^4}{(-d)^2}$$

which is not a square; nor are $c \cdot d(F \mid Q)$ squares for $F = L_2$, K_2, N_2 because $D^2 + 2d = 2$ and $D^2 + 4d = 2$ would imply $2 \mid D$.

(d2) If $d = -1$ and $D \neq 3$, we have $\theta = \alpha$, and therefore

$$c = \frac{D}{-d} T_{L_4/Q}(D + \omega)(\alpha + 1) = \frac{D}{-d}(2D(\alpha + \beta + 2)) = D^2 \cdot \frac{2(D + 2)}{-d}$$

which cannot be a square. Also cannot $c \cdot d(F \mid Q)$ be squares for $F = L_2$, K_2, N_2 because $D^2 - 2 = 2(D + 2)$ and $(D^2 - 2)(D^2 - 4) = 2(D + 2)$ have no integral solutions, and $D^2 - 4 = 2(D + 2)$ leads to $D = 4$, in which case M is not square free.

Altogether we get

THEOREM 7. *Let $M = D^2 + 4d > 1$ and $M_2 = D^2 + 2d > 1$ be square free, $d \mid D$ and $D > 0$. Then*

$$E(L_4) = \langle -1, e, u, \theta \rangle = \langle -1, e^\sigma, u^\sigma, \theta^\sigma \rangle = \left\langle -1, \frac{\omega - \alpha}{\beta}, \frac{\omega - \beta}{\alpha}, \theta \right\rangle,$$

where

$$\theta = \eta = \frac{\alpha}{\beta} \qquad \text{if } d \neq \pm 1 \text{ and } (D, d) \neq (5, -5)$$

$$\theta = \sqrt{\frac{\alpha}{-\beta}} = \frac{1 + \sqrt{5}}{2} \qquad \text{if } (D, d) = (5, -5)$$

$$\theta = \alpha \qquad \text{if } d = \pm 1 \text{ and } (D, d) \neq (3, -1)$$

$$\theta = \sqrt{\alpha} = \frac{1 + \sqrt{5}}{2} \qquad \text{if } (D, d) = (3, -1).$$

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